

# Cubic alternating harmonic number sums

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**Abstract** A recent paper of A. Sofo proves some results about sums of products of quadratic alternating harmonic numbers and reciprocal binomial coefficients. In this paper, we extend his result to cubic alternating harmonic number sums and develop new closed form representations of sums of cubic alternating harmonic numbers and reciprocal binomial coefficients. Some interesting (known or new) illustrative special cases as well as immediate consequences of the main results are also considered.

**Keywords** Harmonic numbers; Riemann zeta functions; Binomial coefficients; multiple harmonic (star) sums; polylogarithm functions.

**AMS Subject Classifications (2010):** 05A10; 05A19; 11B65; 11M06; 11M32

## 1 Introduction

In a recent paper [12], A. Sofo prove some results on sums of products of alternating quadratic harmonic numbers and reciprocal binomial coefficients of the form

$$\overline{W}_k(1, 1; p) := \sum_{n=1}^{\infty} \frac{H_n^2}{n^p \binom{n+k}{k}} (-1)^{n+1} \quad (k \in \mathbb{N}) \quad (1.1)$$

for  $p = 0$  or  $1$ . The generalized  $n$ -th harmonic number of order  $k$ ,  $H_n^{(k)}$ , is defined for positive integers  $n$  and  $r$  as ([2, 3, 23, 24])

$$H_n^{(k)} := \sum_{j=1}^n \frac{1}{j^k} \quad (n, k \in \mathbb{N}) \quad (1.2)$$

where the empty sum  $H_0^{(m)}$  is conventionally understood to be zero, and  $H_n := H_n^{(1)}$ . In this paper we will develop identities, closed form representations of alternating quadratic and cubic harmonic numbers and reciprocal binomial coefficients of the form:

$$\overline{W}_k(1, 2; p) := \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^p \binom{n+k}{k}} (-1)^{n+1} \text{ and } \overline{W}_k(\{1\}_3; p) := \sum_{n=1}^{\infty} \frac{H_n^3}{n^p \binom{n+k}{k}} (-1)^{n+1}, \quad (1.3)$$

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for  $p = 0$  and  $1$ . Here the notation  $\{\}_p$  means that the sequence in the bracket is repeated  $p$ -times. The generalized sums of products of alternating harmonic numbers and reciprocal binomial coefficients  $\overline{W}_k(m_1, m_2, \dots, m_r; p)$  are defined by

$$\overline{W}_k(m_1, m_2, \dots, m_r; p) := \sum_{n=1}^{\infty} \frac{H_n^{(m_1)} H_n^{(m_2)} \dots H_n^{(m_r)}}{n^p \binom{n+k}{k}} (-1)^{n+1}, \quad (p \in \mathbb{N}_0, k, r, m_i \in \mathbb{N}), \quad (1.4)$$

where  $\mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ .

While there are many results for sums of harmonic numbers with positive terms. Many harmonic number sums can be expressed in terms of a linear rational combination of classical Riemann zeta values and harmonic numbers. For example we know that [6, 22, 24]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^3}{n^4} &= \frac{231}{16} \zeta(7) - \frac{51}{4} \zeta(3) \zeta(4) + 2 \zeta(2) \zeta(5), \\ \sum_{n=1}^{\infty} \frac{[H_n^{(2)}]^2}{n^5} &= -\frac{1069}{36} \zeta(9) + \frac{4}{3} \zeta^3(3) + 7 \zeta(2) \zeta(7) - \frac{4}{3} \zeta(3) \zeta(6) + \frac{33}{2} \zeta(4) \zeta(5), \\ \sum_{n=1}^{\infty} \frac{[H_n^{(2)}]^2 H_n^{(3)}}{n^2} &= -\frac{617}{72} \zeta(9) + \zeta^3(3) + \frac{91}{8} \zeta(2) \zeta(7) - \frac{17}{4} \zeta(4) \zeta(5) - \frac{329}{84} \zeta(3) \zeta(6) \end{aligned}$$

and from [11, 23]

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n \binom{n+k}{k}} = \sum_{r=1}^k (-1)^{r+1} \binom{k}{r} \left\{ \begin{aligned} &2 \zeta(4) + 2 \zeta(3) H_{r-1} + \frac{1}{2} \zeta(2) H_{r-1}^2 + \sum_{i=1}^{r-1} \frac{H_i}{i^3} \\ &-\frac{1}{2} \zeta(2) H_{r-1}^{(2)} - \frac{1}{2} \sum_{i=1}^{r-1} \frac{H_i^2 + H_i^{(2)}}{i^2} - \sum_{i=1}^{r-1} \frac{1}{i} \sum_{j=1}^i \frac{H_j}{j^2} \end{aligned} \right\},$$

there are fewer results for sums of the type (1.4). Here the classical Riemann zeta function defined by ([2, 3, 20])

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \Re(p) > 1. \quad (1.5)$$

Some results for sums of (alternating) harmonic numbers may be seen in the works of [1, 4, 5, 7, 9, 10, 12–16, 19–22] and references therein. Some explicit, and closely related results may also be seen in the well presented papers [11, 17, 23]. For example, in [23], we give explicit formulas for the following type of Euler sums function

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^p \binom{n+k}{k}} x^n, \sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n^p \binom{n+k}{k}} x^n, \quad x \in [-1, 0) \cup (0, 1)$$

by using the method of partial fraction decomposition and integral representations of series. The purpose of the present paper is to establish closed form of harmonic number sums (1.3).

Next, we begin with some basic notation. For  $k \in \mathbb{N}$ ,  $\mathbf{s} = (s_1, \dots, s_k) \in (\mathbb{Z}^*)^k$  ( $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \dots\}$ ), and a non-negative integer  $n$ , the multiple harmonic star sum is defined by ([23])

$$\zeta_n^*(\mathbf{s}) \equiv \zeta_n^*(s_1, \dots, s_k) := \sum_{n \geq n_1 \geq \dots \geq n_k \geq 1} \prod_{j=1}^k n_j^{-|s_j|} \text{sgn}(s_j)^{n_j}. \quad (1.6)$$

Throughout the paper we will use  $\bar{n}$  to denote a negative entry  $s_j = -n$ . For example,

$$\zeta_n^*(\bar{2}) = \zeta_n^*(-2), \zeta_n^*(3, \bar{1}) = \zeta_n^*(3, -1), \zeta_n^*(\bar{2}, 2, \bar{1}) = \zeta_n^*(-2, 2, -1).$$

We call  $l(\mathbf{s}) := k$  the depth of (1.6) and  $|\mathbf{s}| := \sum_{j=1}^k |s_j|$  the weight. For convenience we set  $\zeta_n^*(\emptyset) = 1$  and  $\{s_1, \dots, s_j\}_d$  the set formed by repeating the composition  $(s_1, \dots, s_j)$   $d$  times. When taking the limit  $n \rightarrow \infty$  we get the so-called the star Euler sum

$$\zeta^*(\mathbf{s}) = \lim_{n \rightarrow \infty} \zeta_n^*(\mathbf{s}).$$

When  $\mathbf{s} \in \mathbb{N}^k$  they are called the multiple zeta star value. It is obvious that

$$\zeta_n^*(m) = H_n^{(m)}, m \in \mathbb{N}.$$

In this paper, we will prove that the alternating quadratic and cubic harmonic number sums  $\overline{W}_k(1, 2; p)$  and  $\overline{W}_k(\{1\}_3; p)$  for  $p = 0, 1$  can be expressed as a rational linear combination of products of single zeta values and multiple harmonic star sum of weight  $\leq 4$  and depth  $\leq 4$ . The main results of this paper as follow.

**Theorem 1.1** *For positive integer  $k$ , then the following identities hold:*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n+k} (-1)^{n+k} &= -\frac{5}{16} \zeta(4) - \frac{1}{4} \zeta(2) \ln^2 2 + \frac{7}{8} \zeta(3) \ln 2 + \frac{7}{8} \zeta(3) \zeta_{k-1}^*(\bar{1}) \\ &\quad - \frac{1}{4} \zeta(3) H_{k-1} - \frac{1}{2} \zeta(2) \zeta_{k-1}^*(\bar{2}) - \zeta_{k-1}^*(3, \bar{1}) + \zeta_{k-1}^*(1, 2, \bar{1}) \\ &\quad + \frac{1}{2} \zeta(2) \zeta_{k-1}^*(1, \bar{1}) + \zeta_{k-1}^*(2, 1, \bar{1}) + \ln 2 \{ \zeta_{k-1}^*(\bar{3}) - \zeta_{k-1}^*(3) \} \\ &\quad - \frac{1}{2} \ln 2 \zeta(2) \{ \zeta_{k-1}^*(\bar{1}) - \zeta_{k-1}^*(1) \} + \frac{1}{2} \ln^2 2 \{ \zeta_{k-1}^*(\bar{2}) - \zeta_{k-1}^*(2) \} \\ &\quad - \ln 2 \{ \zeta_{k-1}^*(2, \bar{1}) - \zeta_{k-1}^*(2, 1) \} - \ln 2 \{ \zeta_{k-1}^*(1, \bar{2}) - \zeta_{k-1}^*(1, 2) \}, \quad (1.7) \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^3}{n+k} (-1)^{n+k} &= -\frac{5}{16} \zeta(4) + \frac{9}{8} \zeta(3) \ln 2 - \frac{3}{4} \zeta(2) \ln^2 2 + \frac{1}{4} \ln^4 2 + \frac{9}{8} \zeta(3) \zeta_{k-1}^*(\bar{1}) \\ &\quad - \frac{1}{2} \zeta(2) \zeta_{k-1}^*(\bar{2}) - \zeta_{k-1}^*(3, \bar{1}) + \frac{3}{2} \zeta(2) \zeta_{k-1}^*(1, \bar{1}) - \frac{3}{4} \zeta(3) H_{k-1} \\ &\quad + 3 \zeta_{k-1}^*(1, 2, \bar{1}) + 3 \zeta_{k-1}^*(2, 1, \bar{1}) - 6 \zeta_{k-1}^*(\{1\}_3, \bar{1}) \\ &\quad + \ln 2 \{ \zeta_{k-1}^*(\bar{3}) - \zeta_{k-1}^*(3) \} - 3 \ln 2 \{ \zeta_{k-1}^*(2, \bar{1}) - \zeta_{k-1}^*(2, 1) \} \\ &\quad - 3 \ln 2 \{ \zeta_{k-1}^*(1, \bar{2}) - \zeta_{k-1}^*(1, 2) \} + 6 \ln 2 \{ \zeta_{k-1}^*(1, 1, \bar{1}) - \zeta_{k-1}^*(\{1\}_3) \} \\ &\quad - 3 \ln^2 2 \{ \zeta_{k-1}^*(1, \bar{1}) - \zeta_{k-1}^*(1, 1) \} + \frac{3}{2} \ln^2 2 \{ \zeta_{k-1}^*(\bar{2}) - \zeta_{k-1}^*(2) \} \\ &\quad + \left\{ \ln^3 2 - \frac{3}{2} \ln 2 \zeta(2) \right\} \{ \zeta_{k-1}^*(\bar{1}) - \zeta_{k-1}^*(1) \}. \quad (1.8) \end{aligned}$$

We will prove Theorem 1.1 in section 3.

**Theorem 1.2** *For integer  $k \in \mathbb{N}$ , we have*

$$\begin{aligned}\overline{W}_k(1, 2; 0) &= \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n+r} (-1)^{n+1}, \\ \overline{W}_k(\{1\}_3; 0) &= \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} \sum_{n=1}^{\infty} \frac{H_n^3}{n+r} (-1)^{n+1}, \\ \overline{W}_k(1, 2; 1) &= \sum_{r=1}^k (-1)^{r+1} \binom{k}{r} \left\{ \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n} (-1)^{n+1} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n+r} (-1)^{n+1} \right\}, \\ \overline{W}_k(\{1\}_3; 1) &= \sum_{r=1}^k (-1)^{r+1} \binom{k}{r} \left\{ \sum_{n=1}^{\infty} \frac{H_n^3}{n} (-1)^{n+1} - \sum_{n=1}^{\infty} \frac{H_n^3}{n+r} (-1)^{n+1} \right\}.\end{aligned}$$

*Proof.* We consider the expansion

$$\frac{1}{\prod_{i=1}^k (n + a_i)} = \sum_{j=1}^k \frac{A_j}{n + a_j} \quad (k \in \mathbb{N}_0; a_i \in \mathbb{C} \setminus \mathbb{Z}^-) \quad (1.9)$$

where

$$A_j = \lim_{n \rightarrow -a_j} \frac{n + a_j}{\prod_{i=1}^k (n + a_i)} = \prod_{i=1, i \neq j}^k (a_i - a_j)^{-1}. \quad (1.10)$$

Taking  $a_i = a + i$  in (1.10) we obtain

$$A_r = (-1)^{r+1} \frac{r}{k!} \binom{k}{r}, \quad (1.11)$$

$$\frac{1}{\prod_{i=1}^k (n + a + i)} = \sum_{r=1}^k (-1)^{r+1} \frac{r}{k!} \binom{k}{r} \frac{1}{n + a + r}. \quad (1.12)$$

Furthermore, by using the equation (1.12) and the definition of binomial coefficient, letting  $a = 0$ , we have the following expansions

$$\frac{1}{\binom{n+k}{k}} = \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} \frac{1}{n+r} \quad (k \in \mathbb{N}_0), \quad (1.13)$$

Hence, by a direct calculation we may easily deduce the desired result. This completes the proof of Theorem 1.2.  $\square$

## 2 Some lemmas and theorems

The following lemma will be useful in the development of the main theorem 1.1.

**Lemma 2.1** ([23]) *For integers  $m, k \in \mathbb{N}$  and  $x \in [-1, 1)$ , we have*

$$\begin{aligned} & (-1)^m m! \sum_{n \geq m} \frac{s(n+1, m+1)}{(n+k)n!} x^{n+k} + \frac{1}{m+1} \ln^{m+1}(1-x) \\ &= \sum_{j=1}^{m-1} (-1)^{j-1} j! \binom{m}{j} \ln^{m-j}(1-x) \left( \zeta_{k-1}^* \left( \{1\}_{j+1}; \{1\}_j, x \right) - \zeta_{k-1}^* \left( \{1\}_{j+1} \right) \right) \\ & \quad - \ln^m(1-x) \left( \zeta_{k-1}^* (1; x) - \zeta_{k-1}^* (1) \right) - (-1)^m m! \zeta_{k-1}^* \left( \{1\}_{m+1}; \{1\}_m, x \right). \end{aligned} \quad (3.1)$$

where  $s(n, k)$  denotes the (unsigned) Stirling number of the first kind ([8]),

$$\begin{aligned} s(n, 1) &= (n-1)!, s(n, 2) = (n-1)! H_{n-1}, s(n, 3) = \frac{(n-1)!}{2} [H_{n-1}^2 - H_{n-1}^{(2)}], \\ s(n, 4) &= \frac{(n-1)!}{6} [H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}], \\ s(n, 5) &= \frac{(n-1)!}{24} [H_{n-1}^4 - 6H_{n-1}^{(4)} - 6H_{n-1}^2H_{n-1}^{(2)} + 3(H_{n-1}^{(2)})^2 + 8H_{n-1}H_{n-1}^{(3)}]. \end{aligned}$$

The Stirling numbers  $s(n, k)$  of the first kind satisfy a recurrence relation in the form

$$s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k), \quad n, k \in \mathbb{N},$$

with  $s(n, k) = 0, n < k, s(n, 0) = s(0, k) = 0, s(0, 0) = 1$ . The generating function of  $s(n, k)$  is

$$\ln^k(1-x) = (-1)^k k! \sum_{n=k}^{\infty} \frac{s(n, k)}{n!} x^n, \quad -1 \leq x < 1. \quad (3.2)$$

For  $s_j > 0$ ,  $\zeta_n^*(s_1, s_2, \dots, s_m; x_1, x_2, \dots, x_m)$  denotes the partial sums of multiple polylogarithm-star function defined by ([23])

$$\zeta_n^*(s_1, s_2, \dots, s_m; x_1, x_2, \dots, x_m) = \sum_{1 \leq k_m \leq \dots \leq k_1 \leq n} \prod_{j=1}^m \frac{x_j^{k_j}}{k_j^{s_j}}.$$

**Lemma 2.2** ([23]) *For integers  $m, k \in \mathbb{N}$  and  $x \in [-1, 1)$ , we have*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n+k} x^{n+k} &= \sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n} x^n - \text{Li}_{m+1}(x) - \sum_{j=1}^{m-1} (-1)^{j-1} \text{Li}_{m+1-j}(x) \zeta_{k-1}^*(j; x) \\ & \quad - (-1)^m \ln(1-x) \{ \zeta_{k-1}^*(m; x) - \zeta_{k-1}^*(m) \} + (-1)^m \zeta_{k-1}^*(m, 1; 1, x), \end{aligned} \quad (3.3)$$

where  $\text{Li}_p(x)$  denotes the polylogarithm function defined for  $|x| \leq 1$  by the series

$$\text{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \quad \Re(p) > 1. \quad (3.4)$$

**Lemma 2.3** ([11, 22]) For integers  $n \geq 1, k \geq 0$ , then

$$\int_0^1 t^{n-1} \ln^k (1-t) dt = (-1)^k \frac{Y_k(n)}{n}, Y_0(n) = 1, \quad (3.5)$$

where  $Y_k(n) = Y_k(H_n, 1!H_n^{(2)}, 2!H_n^{(3)}, \dots, (r-1)!H_n^{(r)}, \dots)$ ,  $Y_k(x_1, x_2, \dots)$  stands for the complete exponential Bell polynomial defined by (see [8])

$$\exp \left( \sum_{m \geq 1} x_m \frac{t^m}{m!} \right) = 1 + \sum_{k \geq 1} Y_k(x_1, x_2, \dots) \frac{t^k}{k!}. \quad (3.6)$$

From the definition of the complete exponential Bell polynomial, we have

$$\begin{aligned} Y_1(n) &= H_n, Y_2(n) = H_n^2 + H_n^{(2)}, Y_3(n) = H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}, \\ Y_4(n) &= H_n^4 + 8H_n H_n^{(3)} + 6H_n^2 H_n^{(2)} + 3(H_n^{(2)})^2 + 6H_n^{(4)}. \end{aligned}$$

**Lemma 2.4** ([11]) For integer  $m \geq 1$ , and  $-1 < x < 1$ , then

$$\sum_{n=1}^{\infty} H_n H_n^{(m)} x^n = \frac{1}{1-x} \left\{ \sum_{n=1}^{\infty} \frac{H_n}{n^m} x^n - \sum_{n=1}^{\infty} \frac{1}{n^m} \left( \sum_{k=1}^n \frac{x^k}{k} \right) - \zeta(m) \ln(1-x) \right\}. \quad (3.7)$$

**Theorem 2.5** For any real  $x \in (-1, 1)$ , then the following identity holds:

$$\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n = \frac{1}{1-x} \left\{ 2\text{Li}_3(x) - \ln(1-x) \text{Li}_2(x) - \sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n \right\}. \quad (3.8)$$

*Proof.* To prove identity (3.8), we consider the nested sum

$$\sum_{n=1}^{\infty} \frac{y^n}{n^m} \left( \sum_{k=1}^n \frac{x^k}{k^p} \right), \quad x, y \in [-1, 1), \quad m, p \in \mathbb{N}.$$

By taking the sum over complementary pairs of summation indices, we obtain a simple reflection formula

$$\sum_{n=1}^{\infty} \frac{y^n}{n^m} \left( \sum_{k=1}^n \frac{x^k}{k^p} \right) + \sum_{n=1}^{\infty} \frac{x^n}{n^p} \left( \sum_{k=1}^n \frac{y^k}{k^m} \right) = \text{Li}_p(x) \text{Li}_m(y) + \text{Li}_{p+m}(xy). \quad (3.9)$$

Setting  $p = 1, m = 2, y = 1$  in above equation we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sum_{k=1}^n \frac{x^k}{k} \right) + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n} x^n = -\ln(1-x) \zeta(2) + \text{Li}_3(x). \quad (3.10)$$

On the other hand, by the definition of polylogarithm function and Cauchy product of power series, we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n} x^n = \int_0^x \frac{\text{Li}_2(t)}{t(1-t)} dt = \int_0^x \frac{\text{Li}_2(t)}{t} dt + \int_0^x \frac{\text{Li}_2(t)}{1-t} dt$$

$$\begin{aligned}
&= \text{Li}_3(x) - \ln(1-x) \text{Li}_2(x) - \int_0^x \frac{\ln^2(1-t)}{t} dt \\
&= 3\text{Li}_3(x) - \ln(1-x) \text{Li}_2(x) - 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n.
\end{aligned} \tag{3.11}$$

Then, substituting (3.11) into (3.10) yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sum_{k=1}^n \frac{x^k}{k} \right) = 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n + \ln(1-x) \text{Li}_2(x) - 2\text{Li}_3(x) - \ln(1-x) \zeta(2). \tag{3.12}$$

Hence, taking  $m = 2$  in Lemma 2.4 and combining formula (3.12) we may deduce the desired result. The proof of Theorem 2.5 is thus completed.  $\square$

**Theorem 2.6** *If  $m \geq 1$  is a integer and  $z \in [0, 1]$ , then we have*

$$\begin{aligned}
\int_0^z \frac{\ln^m(1+x)}{x} dx &= \frac{1}{m+1} \ln^{m+1}(1+z) + m! \left( \zeta(m+1) - \text{Li}_{m+1} \left( \frac{1}{1+z} \right) \right) \\
&\quad - m! \sum_{j=1}^m \frac{\ln^{m-j+1}(1+z)}{(m-j+1)!} \text{Li}_j \left( \frac{1}{1+z} \right).
\end{aligned} \tag{3.13}$$

*Proof.* We note that the integral in (3.13) can be rewritten as

$$\begin{aligned}
\int_0^z \frac{\ln^m(1+x)}{x} dx &\stackrel{t=1+x}{=} \int_1^{1+z} \frac{\ln^m t}{t-1} dt \stackrel{u=t^{-1}}{=} (-1)^{m+1} \int_1^{(1+z)^{-1}} \frac{\ln^m u}{u-u^2} du \\
&= (-1)^{m+1} \left\{ \int_1^{(1+z)^{-1}} \frac{\ln^m u}{u} du + \int_1^{(1+z)^{-1}} \frac{\ln^m u}{1-u} du \right\} \\
&= \frac{1}{m+1} \ln^{m+1}(1+z) + (-1)^{m+1} \int_1^{(1+z)^{-1}} \frac{\ln^m u}{1-u} du.
\end{aligned} \tag{3.14}$$

We use the elementary integral identity

$$\int t^{n-1} (\ln t)^m dt = t^n \left\{ \frac{\ln^m t}{n} - \sum_{l=1}^m \frac{(-1)^{l+1} \ln^{m-l}(t) (m)_l}{n^{l+1}} \right\}, \tag{3.15}$$

Here  $(m)_l = m(m-1) \cdots (m-l+1)$ . Then yields

$$\begin{aligned}
\int_1^{(1+z)^{-1}} \frac{\ln^m u}{1-u} du &= \sum_{n=1}^{\infty} \int_1^{(1+z)^{-1}} u^{n-1} \ln^m u du = (-1)^{m+1} m! \left( \zeta(m+1) - \text{Li}_{m+1}((1+z)^{-1}) \right) \\
&\quad + m! (-1)^m \sum_{j=1}^m \frac{\ln^{m-j+1}(1+z)}{(m-j+1)!} \text{Li}_j((1+z)^{-1}).
\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.14) yields the desired result.  $\square$

Setting  $p = 3, 4$  in (3.13), we get

$$\int_0^1 \frac{\ln^3(1+x)}{x} dx = 6\zeta(4) + \frac{3}{2}\zeta(2)\ln^2 2 - \frac{1}{4}\ln^4 2 - \frac{21}{4}\zeta(3)\ln 2 - 6\text{Li}_4\left(\frac{1}{2}\right),$$

$$\int_0^1 \frac{\ln^4(1+x)}{x} dx = -24\text{Li}_5\left(\frac{1}{2}\right) - 24\ln 2\text{Li}_4\left(\frac{1}{2}\right) - \frac{4}{5}\ln^5 2 - \frac{21}{2}\zeta(3)\ln^2 2 + 24\zeta(5) + 4\zeta(2)\ln^3 2.$$

**Theorem 2.7** For integer  $m \in \mathbb{N}$ , then

$$\sum_{n=1}^{\infty} \frac{Y_m(n)}{n} (-1)^{n-1} = m! \text{Li}_{m+1}\left(\frac{1}{2}\right). \quad (3.17)$$

*Proof.* First, by a direct calculation, we get

$$\int_0^1 \frac{\ln^m(1-x)}{1+x} dx = (-1)^m m! \text{Li}_{m+1}\left(\frac{1}{2}\right). \quad (3.18)$$

Then by applying the known result

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}, \quad x \in (-1, 1),$$

in (3.18), we obtain

$$\begin{aligned} \int_0^1 \frac{\ln^m(1-x)}{1+x} dx &= \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 x^{n-1} \ln^m(1-x) dx \\ &= (-1)^m \sum_{n=1}^{\infty} \frac{Y_m(n)}{n} (-1)^{n-1}. \end{aligned} \quad (3.19)$$

Therefore, combining (3.18) with (3.19), we obtain the formula (3.17). This completes the proof of Theorem 2.7.  $\square$

**Corollary 2.8** The following identities hold:

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n} (-1)^{n-1} = \frac{5}{8}\zeta(4) + \frac{3}{4}\zeta(2)\ln^2 2 - \frac{1}{4}\ln^4 2 - \frac{9}{8}\zeta(3)\ln 2, \quad (3.20)$$

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n} (-1)^{n-1} = 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{12}\ln^4 2 + \frac{7}{8}\zeta(3)\ln 2 - \frac{1}{4}\zeta(2)\ln^2 2 - \zeta(4). \quad (3.21)$$

*Proof.* Taking  $m = 3$  in Theorem 2.6 and Theorem 2.7, we have

$$\sum_{n=1}^{\infty} \frac{H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{n} (-1)^{n-1} = \int_0^1 \frac{\ln^3(1+x)}{x} dx - \frac{1}{4}\ln^4 2, \quad (3.22)$$



$$\sum_{n=1}^{\infty} \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n} (-1)^{n-1} = 6\text{Li}_4\left(\frac{1}{2}\right). \quad (3.23)$$

From [6] we deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \left( \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \right) = \frac{7}{4} \zeta(3) \ln 2 - \frac{5}{16} \zeta(4). \quad (3.24)$$

Then substituting (3.24) into (3.9) with  $m = 3, p = 1, x = -1, y = 1$  yields

$$\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n} (-1)^{n-1} = \frac{19}{16} \zeta(4) - \frac{3}{4} \zeta(3) \ln 2. \quad (3.25)$$

Thus, combining formulas (3.22), (3.23) and (3.25) we obtain the results.  $\square$

**Corollary 2.9** *For positive integer  $k$ , then we have*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{n+k} (-1)^{n+k} &= \frac{1}{4} \ln^4 2 - 6\zeta_{k-1}^* (\{1\}_3, \bar{1}) + \ln^3 2 \{ \zeta_{k-1}^* (\bar{1}) - \zeta_{k-1}^* (1) \} \\ &\quad - 3\ln^2 2 \{ \zeta_{k-1}^* (1, \bar{1}) - \zeta_{k-1}^* (1, 1) \} \\ &\quad + 6\ln 2 \{ \zeta_{k-1}^* (1, 1, \bar{1}) - \zeta_{k-1}^* (\{1\}_3) \}, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n+k} (-1)^{n+k} &= -\frac{5}{16} \zeta(4) + \frac{3}{4} \zeta(3) \ln 2 + \ln 2 \{ \zeta_{k-1}^* (\bar{3}) - \zeta_{k-1}^* (3) \} + \frac{3}{4} \zeta(3) \zeta_{k-1}^* (\bar{1}) \\ &\quad - \frac{1}{2} \zeta(2) \zeta_{k-1}^* (\bar{2}) - \zeta_{k-1}^* (3, \bar{1}). \end{aligned} \quad (3.27)$$

*Proof.* Taking  $m = 3, x = -1$  in Lemma 2.1 and Lemma 2.2 we obtain the result.  $\square$

### 3 Proof of Theorem 1.1

In (3.8), changing  $x$  to  $t$ , then multiplying it by  $t^{k-1} - t^{-1}$  and integrating over  $(0, x)$ , we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \frac{H_n H_n^{(2)}}{n+k} x^{n+k} - \frac{H_n H_n^{(2)}}{n} x^n \right\} \\ &= - \sum_{i=1}^k \int_0^x t^{i-2} \left\{ 2\text{Li}_3(t) - \ln(1-t) \text{Li}_2(t) - \sum_{n=1}^{\infty} \frac{H_n}{n^2} t^n \right\} dt \\ &= - \left\{ 2 \int_0^x \frac{\text{Li}_3(t)}{t} dt - \int_0^x \frac{\ln(1-t) \text{Li}_2(t)}{t} dt - \sum_{n=1}^{\infty} \frac{H_n}{n^2} \int_0^x t^{n-1} dt \right\} \\ &\quad - \sum_{i=1}^{k-1} \int_0^x t^{i-1} \left\{ 2\text{Li}_3(t) - \ln(1-t) \text{Li}_2(t) - \sum_{n=1}^{\infty} \frac{H_n}{n^2} t^n \right\} dt \end{aligned}$$

$$\begin{aligned}
&= -2\text{Li}_4(x) - \frac{1}{2}\text{Li}_2^2(x) + \sum_{n=1}^{\infty} \frac{H_n}{n^3} x^n \\
&+ \sum_{i=1}^{k-1} \int_0^x t^{i-1} \ln(1-t) \text{Li}_2(t) dt + \sum_{i=1}^{k-1} \sum_{n=1}^{\infty} \frac{H_n}{n^2(n+i)} x^{n+i} \\
&- 2 \sum_{i=1}^{k-1} \left\{ \frac{x^i}{i} \text{Li}_3(x) - \frac{x^i}{i^2} \text{Li}_2(x) - \frac{1}{i^3} \ln(1-x) (x^i - 1) + \frac{1}{i^3} \sum_{j=1}^i \frac{x^j}{j} \right\}. \tag{4.1}
\end{aligned}$$

In [23], we deduce the following identity

$$\int_0^x t^{n-1} \ln(1-t) dt = \frac{1}{n} \left\{ x^n \ln(1-x) - \sum_{j=1}^n \frac{x^j}{j} - \ln(1-x) \right\}. \tag{4.2}$$

Hence, by using (4.2) with the help of Theorem 2.2 in the reference [23], then

$$\begin{aligned}
&\int_0^x t^{i-1} \ln(1-t) \text{Li}_2(t) dt = \int_0^x \text{Li}_2(t) d \left\{ \frac{1}{i} \left[ (t^i - 1) \ln(1-t) - \sum_{j=1}^i \frac{t^j}{j} \right] \right\} \\
&= \frac{1}{i} \left\{ (x^i - 1) \ln(1-x) - \sum_{j=1}^i \frac{x^j}{j} \right\} \text{Li}_2(x) + \frac{1}{i} \int_0^x t^{i-1} \ln^2(1-t) dt \\
&- \frac{1}{i} \int_0^x \frac{\ln^2(1-t)}{t} dt - \frac{1}{i} \sum_{j=1}^i \frac{1}{j} \int_0^x t^{j-1} \ln(1-t) dt \\
&= \frac{1}{i} \left\{ (x^i - 1) \ln(1-x) - \zeta_i^*(1; x) \right\} \text{Li}_2(x) + \frac{1}{i^2} \ln^2(1-x) (x^i - 1) \\
&- \frac{2}{i^2} \ln(1-x) \{ \zeta_i^*(1; x) - \zeta_i^*(1) \} + \frac{2}{i^2} \zeta_i^*(1, 1; 1, x) \\
&- \frac{1}{i} \ln(1-x) \{ \{ \zeta_i^*(2; x) - \zeta_i^*(2) \} \} + \frac{1}{i} \zeta_i^*(2, 1; 1, x) \\
&- \frac{2}{i} \left\{ \sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n - \text{Li}_3(x) \right\}. \tag{4.3}
\end{aligned}$$

Moreover, by a direct calculation we obtain

$$\begin{aligned}
&\sum_{i=1}^{k-1} \sum_{n=1}^{\infty} \frac{H_n}{n^2(n+i)} (-1)^{n+i} = \sum_{i=1}^{k-1} (-1)^{i-1} \sum_{n=1}^{\infty} H_n (-1)^{n-1} \left\{ \frac{1}{i} \cdot \frac{1}{n^2} - \frac{1}{i^2} \cdot \frac{1}{n} + \frac{1}{i^2} \cdot \frac{1}{n+i} \right\} \\
&= \sum_{i=1}^{k-1} \frac{(-1)^{i-1}}{i} \sum_{n=1}^{\infty} \frac{H_n}{n^2} (-1)^{n-1} - \sum_{i=1}^{k-1} \frac{(-1)^{i-1}}{i^2} \sum_{n=1}^{\infty} \frac{H_n}{n} (-1)^{n-1} + \sum_{i=1}^{k-1} \frac{(-1)^{i-1}}{i^2} \sum_{n=1}^{\infty} \frac{H_n}{n+i} (-1)^{n-1} \\
&= -\frac{5}{8} \zeta(3) \zeta_{k-1}^*(\bar{1}) + \frac{\zeta(2) - \ln^2 2}{2} \zeta_{k-1}^*(\bar{2}) + \ln 2 \{ \zeta_{k-1}^*(2, \bar{1}) - \zeta_{k-1}^*(2, 1) \} \\
&- \ln 2 \{ \zeta_{k-1}^*(\bar{3}) - \zeta_{k-1}^*(3) \} + \frac{1}{2} \ln^2 2 \zeta_{k-1}^*(\bar{2}) - \zeta_{k-1}^*(2, 1, \bar{1}) + \zeta_{k-1}^*(3, \bar{1}). \tag{4.4}
\end{aligned}$$

Hence, letting  $x = -1$  in (4.1), (4.3) and combining identities (3.21), (3.26), (3.27) and (4.4), and using the following formula ([6])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} (-1)^{n-1} = -2\text{Li}_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - \frac{7}{4}\zeta(3)\ln 2,$$

by a simple calculation, we can prove the Theorem 1.1. □

Taking  $k = 2$  in Theorem 1.1, we can get the following results.

**Corollary 3.1** *The following identities hold:*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^3}{n+2} (-1)^n &= -\frac{5}{16}\zeta(4) + \frac{9}{8}\zeta(3)\ln 2 - \frac{3}{4}\zeta(2)\ln^2 2 + \frac{1}{4}\ln^4 2 - 2\ln 2 \\ &\quad + 3\ln^2 2 - 2\ln^3 2 + 3\zeta(2)\ln 2 - \frac{15}{8}\zeta(3) - \zeta(2) + 1, \\ \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n+2} (-1)^n &= -\frac{5}{16}\zeta(4) - \frac{1}{4}\zeta(2)\ln^2 2 + \frac{7}{8}\zeta(3)\ln 2 \\ &\quad - \frac{9}{8}\zeta(3) + \zeta(2)\ln 2 + 2\ln 2 - \ln^2 2 - 1. \end{aligned}$$

## 4 Conclusion

From Theorem 1.1, Theorem 1.2 and Corollary 2.8, we obtain the following description.

**Theorem 4.1** *For  $p = 0, 1$ , then the alternating quadratic and cubic harmonic number sums  $\overline{W}_k(1, 2; p)$  and  $\overline{W}_k(\{1\}_3; p)$  can be expressed as a rational linear combination of products of single zeta values and multiple harmonic star sum of weight  $\leq 4$  and depth  $\leq 4$ .*

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## References

- [1] David H. Bailey, Jonathan M. Borwein and Roland Girgensohn. *Experimental evaluation of Euler sums*. Experimental Mathematics., 1994, **3**(1): 17-30.
- [2] B. C. Berndt. *Ramanujans Notebooks, Part I*. Springer-Verlag, New York., 1985.
- [3] B. C. Berndt. *Ramanujans Notebooks, Part II*. Springer-Verlag, New York., 1989.
- [4] David Borwein, Jonathan M. Borwein and Roland Girgensohn. *Explicit evaluation of Euler sums*. Proc. Edinburgh Math., 1995, **38**: 277-294.
- [5] J. M. Borwein, I. J. Zucker, J. Boersma. *The evaluation of character Euler double sums*. Ramanujan J., 2008, **15** (3): 377-405.
- [6] Philippe Flajolet and Bruno Salvy. *Euler sums and contour integral representations*. Experimental Mathematics., 1998, **7**(1): 15-35.
- [7] Pedro Freitas. *Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums*. Mathematics of Computation., 2005, **74**(251): 1425-1440.
- [8] Comtet L. Advanced combinatorics, Boston: D Reidel Publishing Company, 1974.

- [9] I. Mezö. *Nonlinear Euler sums*. Pacific J. Math., 2014, **272**: 201-226.
- [10] Eie, Minking, W. C. Liaw. *Double Euler sums on Hurwitz Zeta functions*. Rocky Mountain Journal of Mathematics., 2009, **39**: 1869-1883.
- [11] Xin Si, Ce Xu, Mingyu Zhang. *Quadratic and cubic harmonic number sums*. J. Math. Anal. Appl., 2017, **447**: 419-434.
- [12] A. Sofo. *Quadratic alternating harmonic number sums*. J. Number Theory., 2015, **154**: 144-159.
- [13] A. Sofo. *Integral identities for sums*. Math. Commun., 2008, **13**: 303-309.
- [14] A. Sofo. *Harmonic sums and integral representations*. J. Appl. Anal., 2010, **16**: 265-277.
- [15] A. Sofo. *Harmonic number sums in closed form*. Math. Commun., 2011, **16**: 335-345.
- [16] A. Sofo. *Shifted harmonic sums of order two*. Commun. Korean Math. Soc., 2014, **29**(2): 239-255.
- [17] A. Sofo. *Harmonic numbers at half integer values*. Integral Transforms and Special Functions., 2016, **27**(6): 430-442.
- [18] A. Sofo, D. Cvijović. *Extensions of Euler harmonic sums*. Appl. Anal. Discrete Math. 2012, **6**: 317-328.
- [19] A. Sofo, H.M. Srivastava. *Identities for the harmonic numbers and binomial coefficients*. Ramanujan J., 2011, **25**: 93-113.
- [20] H. M. Srivastava, J. Choi. *Zeta and q-Zeta Functions and Associated Series and Integrals*. Elsevier Science Publishers., Amsterdam, London and New York, 2012.
- [21] Ce Xu, Jinfa Cheng. *Some results on Euler sums*. Functions et Approximatio., 2016, **54**(1): 25-37.
- [22] Ce Xu, Yuhuan Yan, Zhijuan Shi. *Euler sums and integrals of polylogarithm functions*. J. Number Theory., 2016, **165**: 84-108.
- [23] Ce Xu, Mingyu Zhang, Weixia Zhu. *Some evaluation of harmonic number sums*. Integral Transforms and Special Functions., 2016, **27**(12): 937-955.
- [24] Ce Xu, Zhonghua Li. *Tornheim type series and nonlinear Euler sums*. J. Number Theory., 2017, **174**: 40-67.